SOLUTION OF INVERSE PROBLEMS WITH AN UNKNOWN MODEL OF THE PROCESS

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A method is proposed for identifying systems with distributed parameters in the case of no a priori information about the form of an adequate model of the process being studied.

The development of processing methods that would permit taking account of physical features of the process under investigation is urgent for the theory of interpretation of a thermophysical experiment. Consequently, the mathematical modeling of the results of an experiment by using inverse problems [1-4] in which it is required to restore its parameters by certain observations on the solution of a given equation, is of interest. The distinction of the approach based on solving inverse problems from other methods of interpretation [5-7] is the passage from mathematical models, of regression type, say, in which the conceptual structure and the functional connections of the effective factors are not reflected, to models in the form of boundary value problems of mathematical physics, say, that contain information about the nature of the processes being observed. Combining experiment with the solution of boundary value problems permits taking account of the physical features of the object and to perform modeling in which not experimental data, and not data stacked within the framework of the model selected, but the model itself will be subjected to formalization.

The theory of inverse problems developed at this time requires the assignment of an adequatel model of the process being investigated. On the basis of the general theory of incorrectly posed problems [8, 9], it turns out to be possible to eliminate the postulation of the final form of the mathematical model by setting a certain one of its set in conformity with the process under consideration. Indeed, in practice the necessary information about the object under investigation is often available in the form of the assumed class of mathematical models by which the effective factors their structure and interrelation are reflected. Then a solution can be sought, which satisfies the boundary value problem under consideration on the one hand, and is consistent with given observations on the other. The degree of consistency in this case is the criterion by which satisfaction of the necessary conditions for adequacy between the model and the process is judged. The passage to another model should evidently be pursued when consistency is not satisfied. The structure and significance of the effective factors can be clarified as a result of a sequential choice and comparative analysis of the solutions obtained.

From the viewpoint of systems theory [10], formulation of inverse problems in a set of given models permits simultaneous structural and parametric identification of the object. The requirement of sequential solution of a whole series of inverse problems with the utilization of the identical initial data distinguishes it from other formulations.

As is known [8, 9], the initial hypothesis for the solution of incorrect problems is the narrowing of the total class of possible states to boundaries that do not spoil consistency with the initial data. This principle should also be used in selecting the sequence of inverse problem formulations. Depending on the quantity of initial information, the necessary narrowing can be performed on the basis of different methods by making a selection, in particular, between inductive or deductive modeling.

In the case of giving incomplete a priori information about the form of the model, the functional properties of its parameters, and the limited quantity of samples of observations, inductive modeling has definite advantages. The initial selection of a particular model based on the reflection of the most essential aspects of the process under investigation is a significant constraint on the class of possible states of the object which

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cannot, at the same time, spoil the adequacy conditions. In the opposite case, when the model turns out to be inadequate, the utilization of particular forms should result in unsatisfactory matching with the observations. Hence, the passage from the particular to the volume, which is a result of the inadequacy of the model or the insufficiency of the functional description of the properties of the parameters desired, can eliminate finding arbitrary solutions by which the given sample of observations is described.

If a sufficient quantity of information is given about the form of the adequate model, the properties of its parameters, and the corresponding amount of observations by which identifiability of the selected model is assured, the utilization of deductive modeling turns out to be effective. It consists in going over from the general functional representation of the object to its particular form.

Below we show the application of the described approach in solving the following temperature diagnostics problem.

Results of measuring the temperature in a copper cylinder of length L = 0.046 m and diameter d = 0.0254 m are presented in [11]. The specimen was heated initially and then cooled in open air. It was determined by measurements of the temperature of the environment that this quantity is constant and equals $u_{av} = 300.6°K$.

Let us identify the parameters of the test object by this quantity of information, and let us also estimate the nonuniformity of the distribution of its temperature field.

We seek the solution of the problem of interest to us in the domain of states reflecting the process of free cooling of a body of high heat conductivity. This permits making the following assumptions. The coefficient of heat elimination is a nonincreasing smooth function. The thermophysical properties are constant in the observed temperature range 320°K < u < 420°K. The thermal resistance of the rod is negligible. Installation of the measurement system can result in local changes in the continuity of the specimen, which are not substantial in the volume of the whole rod.

To strengthen the limitations of the class of possible thermal states of the rod, we perform an initial analysis by using a model with concentrated parameters. Neglecting the thermal resistance, we obtain

$$c\rho \frac{du}{dt} = \frac{2\alpha}{R}(u_{cp} - u), \ t > 0, \ u|_{t=0} = u_0, \tag{1}$$

where $u_0 = 410.7°K$ is the initial value of the temperature being measured.

We seek the heat elimination coefficient $\alpha(t)$ in the form of the function $h = 2\alpha/c\rho R$ for which the polynomial approximation

$$h = \sum_{i=0}^{p} \eta_i t^i \tag{2}$$

can be selected under the assumptions made, where $-\infty < \eta_i < \infty$ are coefficients to be determined.

Taking into account the lack of additional information about the nature of the quantities desired (whereupon extension of the domain of allowable solutions is possible), we seek the unknown parameters by using the method of regularization according to the following scheme:

$$\inf_{a \in A} \Omega[a], \ \max_{1 \leqslant j \leqslant n} |u_j^\delta - u_j| \leqslant \delta, \tag{3}$$

where a are the desired parameters, A is the domain of admissible solutions, Ω is a stabilizing functional, u^δ are observations, u is the solution of the model problem, and $\delta = 0.01$ K is the level of consistency between the observations and the temperatures computed by means of the model, which is selected as the error of the measurements.

The variational formulation (3) permits realization of a limitation on their variations in the case of instability of the identification because of minimization of the functional $\Omega[a]$ which depends on the desired parameters a. As the results [12, 13] show, in selecting a stabilizing functional one should start from the requirement of strengthening the constraints imposed on the domain of allowable solutions of the inverse problem. In this connection, stabilizers of the type of highest order generalized derivatives, allowable by the selected approximation of the desired quantities are used in all the cases considered below.

The stabilizing functional of the above-mentioned type has the form $\Omega = \eta_p^2$ for the function (2). Hence, among the possible values of the coefficient $-\infty < \eta_p < \infty$ a quantity with minimal modulus is selected under the condition of best matching of the model temperature field to the given sample of observations.

Let us note certain questions of the numerical realization of the identification algorithms used. The variational regularization scheme (3) is reduced to an absolute programming problem by the method of penalties. The penalty function has the form

$$F(a) = \Omega[a] + K\gamma(a), \tag{4}$$

where $\gamma = \max_{1 \leqslant j \leqslant n} |u_j^\delta - u_j| - \delta$ is the residual in the consistency conditions, and K is the penalty function ($K \gg 0$ if $\gamma > 0$ but $K = 0$ if $\gamma \leq 0$).

Determination of the coefficients $a = \{\eta_i\}_{i=\overline{0,p}}$ of the model (1) by the method elucidated for p = 2 would permit finding the values $\eta_0 = 2.914\ h^{-1}$, $\eta_1 = -1.52\ h^{-2}$, $\eta_2 = 0.375\ h^{-3}$. The residual in the consistency conditions would reach the value $\gamma_1 = 0.1845°K$. A further successive increase in the degree of the polynomial (2) would show the statistical insignificance of the recurrent change in the form of the desired function h(t) for $p \geq 6$. The form of this function is shown in Fig. 1 for p = 6. The magnitude of the residual is $\gamma_2 = 0.1842°K$ in this case. The form of the desired function, found by means of the finite-difference formula

$$h_j = \frac{u_{j+1}^\delta - u_{j-1}^\delta}{2(t_j - t_{j-1})(u_{av} - u_j^\delta)}, \quad j = \overline{1, n-1},$$

is presented for comparison with the results obtained in Fig. 1.

In order to estimate the level of the identification error and the modeling in the case considered above, we make the passage to the other model that takes account of the radial heat distribution

$$\begin{gathered} c\rho \frac{\partial u}{\partial t} = \frac{\lambda}{R^2} \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right), \quad 0 < r < 1, \quad t > 0; \\ u|_{t=0} = u_0 \equiv \text{const}, \ 0 < r < 1; \\ \alpha(u|_{r=1} - u_{cp}) + \frac{\lambda}{R} \frac{\partial u}{\partial r}\bigg|_{r=1} = 0, \ t > 0. \end{gathered} \tag{5}$$

Since we do not know the values of the coefficients c, ρ, and λ, as well as of the coordinate r_T of the point of temperature measurement u^δ, then we pose the problem of determining the parameters

$$a_1 = \frac{c\rho R^2}{\lambda} \equiv \text{const}, \quad a_2 = \frac{\alpha R}{\lambda}, \quad a_3 = r_T$$

for the model (5). We approximate the coefficient $a_2(t)$ exactly as in the preceding case. We determine the temperature field u(r, t) by Bessel interpolation of the mesh function $w_i^j = u(r_i, t_j)$, $i = \overline{0, 50}$, $j = \overline{0, 100}$ found by the finite-difference method.

With respect to the selection of the model (5) and its desired parameters, we note that expansion of the states, as compared with the model (1), is characterized just by two constant parameters $a_1 > 0$ and $0 \leq a_3 \leq 1$, here conserving the uniform form of the initial temperature distribution over the cylinder section.

The following values of the desired quantities were obtained as a result of identification. The coefficient a_1 turned out to equal 0.000793 h. Verification of this value by using handbook data [14] shows that the difference is 7.2% for c = 3893 J/(kg · °K), ρ = 8800 kg/m^3, λ = 348.8 W/(m · °K). The form of the function $h = 2a_2/a_1$ is shown in Fig. 1. As can be seen, the values found by using models (1) and (5) are in satisfactory mutual agreement. The maximum distinction does not exceed 6% of the running value.

Let us consider the analysis of determining the coordinate of the temperature measurement point in more detail. A numerical investigation of the behavior of the penalty function (4) showed that it is responsive to changes in the parameter a_3. Consequently, the value $a_3 = 0$ was found by a coordinate-by-coordinate descent from the initial value $a_3^{(0)} = 1$. Further variations of the coordinate of the measurement point during the search for the minimum of the penalty function did not exceed the value 10^{-6} m. The final estimate for the nonuniformity of the radial temperature distribution of the rod shows that

$$0.04\text{K} < \operatorname*{var}_{0 \leqslant t \leqslant T} |u(1, t) - u(0, t)| < 0.06\text{K}.$$

For the error level $\delta = 0.01°K$ this quantity confirms the response of the penalty function $F(a)$ to a change in the parameter a_3.

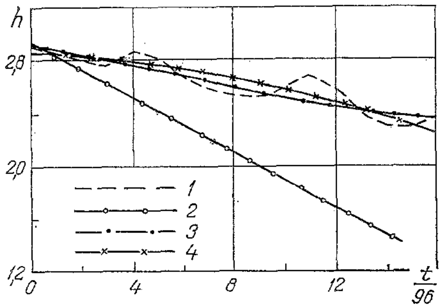

Fig. 1

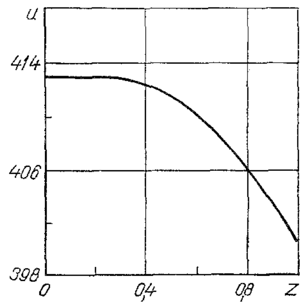

Fig. 2

Fig. 1. Determination of the heat elimination: 1) finite-difference formula; 2) model (1), p = 1; 3) model (1), p = 6; 4) model (5), p = 6. h, h^{-1}; t/96 sec.

Fig. 2. Initial temperature distribution in the segment $0 \le z \le l$, u, K.

Taking account of the thermal radiation

$$\alpha\,(u|_{r=1}-u_{av})+\varepsilon\sigma\,(u^4|_{r=1}-u_{av}^4)+\frac{\lambda}{R}\,\frac{\partial u}{\partial r}\bigg|_{r=1}=0,\quad t>0$$

in the boundary condition showed the lack of significance of this factor.

Therefore, the passage to another model within the framework of the assumed class of test object states confirmed the results obtained earlier and also permitted determination of new parameters and establishment of the significance of such factors as the radial thermal flux and thermal radiation. However, improvement of the consistency with observations as compared with the model (1) was not achieved. The residual turned out to equal $\gamma_3 = 0.2413$°K. Therefore, expansion of the description of the object states reflects the structure of the effective factors insufficiently completely. Since the results obtained are inadequate to the setting up the main source of modeling errors, we continue analyzing the object states by using other models.

Let us examine the boundary value problem

$$\begin{gathered}c\rho\,\frac{\partial u}{\partial t}=\frac{2\lambda}{R^2}\,\frac{\partial^2 u}{\partial\varphi^2}-\frac{2\alpha}{R}\,(u-u_{av}),\ 0<\varphi<2\pi,\ t>0,\\ u|_{t=0}=u_0\equiv\text{const},\ 0<\varphi<2\pi,\\ u|_{\varphi=0}=u|_{\varphi=2\pi},\ \ \frac{\partial u}{\partial\varphi}\bigg|_{\varphi=0}=\frac{\partial u}{\partial\varphi}\bigg|_{\varphi=2\pi},\ \ t>0,\end{gathered}\tag{6}$$

in which the nonuniformity is taken into account in the temperature field distribution over the circular coordinate φ because of the dependence of the heat elimination $\alpha = \alpha(\varphi, t)$. We neglect the radial thermal flux and consider the initial temperature distribution constant. Such a narrowing of the domain of admissible states permits estimation of the significance of the circular nonuniformity of the heat elimination. The location of the measurement system was taken as the origin of circular coordinate measurement, i.e., $\varphi_T = 0$. The quantities $a_1 = c\rho R^2/\lambda \equiv$ const and $\alpha_2 = \alpha R/\lambda$ are the desired parameters. The coefficient a_2 was approximated by the linear polygon function

$$\begin{gathered}a_2(\varphi,\ t)=\frac{1}{4}\,(v_{i-1}^{j-1}+v_{i-1}^{j}+v_i^{j-1}+v_i^{j})+\frac{1}{4}\,(v_i^{j}+v_i^{j-1}-v_{i-1}^{j}-v_{i-1}^{j-1})\,\frac{2\varphi-\varphi_i-\varphi_{i-1}}{\varphi_i-\varphi_{i-1}}+\\ +\frac{1}{4}\,(v_i^{j}-v_i^{j-1}+v_{i-1}^{j}-v_{i-1}^{j-1})\,\frac{2t-t_j-t_{j-1}}{t_j-t_{j-1}}+\\ +\frac{1}{4}\,(v_{i-1}^{j-1}-v_{i-1}^{j}+v_i^{j}-v_i^{j-1})\,\frac{(2\varphi-\varphi_i-\varphi_{i-1})\,(2t-t_j-t_{j-1})}{(\varphi_i-\varphi_{i-1})\,(t_j-t_{j-1})},\\ \varphi_{i-1}\leqslant\varphi\leqslant\varphi_i,\ t_{j-1}\leqslant t\leqslant t_j,\end{gathered}$$

whose nodes $\nu_i^j = a_2(\varphi_i, t_j)$, $i = \overline{0,\,12}$, $j = \overline{0,\,4}$ are to be determined. The stabilizing function was selected in the form

$$\Omega[a_2]=\int_0^{2\pi}\int_0^T\left[\left(\frac{\partial a_2}{\partial\varphi}\right)^2+\left(\frac{\partial a_2}{\partial t}\right)^2\right]d\varphi dt=\sum_i\sum_j\left(f_{ij}^2+\frac{1}{3}g_{ij}^2+\frac{1}{3}q_{ij}^2+s_{ij}^2\right)(\varphi_i-\varphi_{i-1})(t_j-t_{j-1}),$$

where

$$f_{ij}=\frac{v_i^j+v_i^{j-1}-v_{i-1}^j-v_{i-1}^{j-1}}{2(\varphi_i-\varphi_{i-1})};\quad g_{ij}=\frac{v_i^j-v_i^{j-1}-v_{i-1}^j+v_{i-1}^{j-1}}{2(\varphi_i-\varphi_{i-1})};$$

$$q_{ij}=\frac{v_i^j-v_i^{j-1}-v_{i-1}^j+v_{i-1}^{j-1}}{2(t_j-t_{j-1})};\quad s_{ij}=\frac{v_i^j-v_i^{j-1}+v_{i-1}^j-v_{i-1}^{j-1}}{2(t_j-t_{j-1})}.$$

The finite-difference mesh $w_i^j=u(\varphi_i, t_j)$ for solving the direct problem (6) was given with the number of nodes $i=\overline{0, 50}$, $j=\overline{0, 100}$.

Results of the identification $\{u, a_{1,2}\}$ at this stage of the analysis show the negligible deviations of the temperature over the circular coordinate

$$0.001\text{K}<\underset{0\leqslant t\leqslant T}{\text{var}}\,|\max_{0\leqslant\varphi\leqslant 2\pi}u(\varphi,t)-\min_{0\leqslant\varphi\leqslant 2\pi}u(\varphi,t)|<0.005\text{ K},$$

as well as the weak dependence of the heat elimination on the coordinate φ. The magnitude of the residual was $\gamma_4=0.1741$°K. Improvement of the consistency as compared to model (1) is explained by the expansion of the functional representation of the heat elimination since the values of the coefficient a_1 found by using the models (5) and (6) differ insignificantly. This result shows the value of the performable regularization.

Terminating the analysis of the thermal state of the copper rod, by taking account of the preceding results we estimate the nonuniformity of the temperature distribution over the length of the specimen. To do this we consider the model

$$\begin{gathered}c\rho\frac{\partial u}{\partial t}=\frac{\lambda}{l^2}\frac{\partial^2 u}{\partial z^2}-\frac{2\alpha}{R}(u-u_{\text{cp}}),\ 0<z<1,\ t>0,\\ u|_{t=0}=u_0(z),\ 0<z<1,\\ u|_{z=0}=\theta_0(t),\ u|_{z=1}=\theta_1(t),\ t>0.\end{gathered}\tag{7}$$

In this case the domain of allowable states is expanded because of the variable boundary conditions $u_0(z)$ and $\theta_{0,1}(t)$. We seek them in the class of smooth functions. Relative to the thermophysical properties and conditions of the heat elimination, it is assumed that they satisfy the assumptions made for the model (5).

Taking into account the possibility of spoiling the wholeness of the rod in the axial direction because of the disposition of the measurement system, we seek the solution of the inverse problem not over the whole length of the rod but only over a part $l=0.1$L. We consider the ends of the estimation segment equidistant from the temperature measurement points, i.e., $z_T=0.5$. We approximate the unknown functions $u_0(z)$ and $\theta_{0,1}(t)$ by cubic splines [15]. This selection is made with the local properties of splines and the absence of data on the nature of the boundary conditions in the experiment taken into account. We give the stabilizing functional in the form

$$\Omega[u_0,\theta_{0,1}]=\int_0^1\left(\frac{d^3u_0}{dz^3}\right)^2dz+\int_0^T\left[\left(\frac{d^3\theta_0}{dt^3}\right)^2+\left(\frac{d^3\theta_1}{dt^3}\right)^2\right]dt.$$

We determine the temperature field u(z, t) by Bessel interpolation of the mesh function $w_i^j=u(z_i, t_j)$, $i=\overline{0, 50}$, $j=\overline{0, 100}$.

We give the parameters $a_1=c\rho l^2/\lambda$ and $a_2=2\alpha^2 l^2/\lambda R$ from the results obtained by using the model (5), and we determine the boundary conditions $\{u_0, \theta_0, \theta_{0,1}\}$ in the set of smooth positive-definite functions. Reaching the magnitude of the residual $\gamma_5=4.14$°K indicates the inadequacy of the identification in the model (7) of just the boundary condition. Expansion of the number of desired quantities because of the coefficient $a_1=$ const permitted improvement of the consistency to $\gamma_6=0.31$°K. The laws found for the temperature variation are shown in Figs. 2 and 3. We turn attention to the nonuniformity of the initial distribution. Since its limitation by a constant value in the preceding cases permitted satisfactory identification, then extending the model representation of the thermal state of the rod with axial heat conductivity taken into account shows the significance of the change in object properties in this area. This is evidently caused by the method of obtaining the observa-

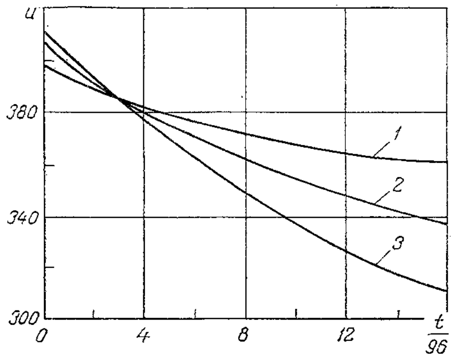

Fig. 3. Rod cooling in the estimation segment $0 \le z \le l$: 1) $u(z, t)|_{z=1}$; 2) $u(z, t)|_{z=z_T}$; 3) $u(z, t)|_{z=0}$.

tion sample. Further detailing of the description of the thermal state should be performed with the properties of the rod and the measurement system taken into account in a complex manner. From this viewpoint, the necessity of a successive increase in the number of desired parameters in model (7) is eliminated because of the additional refinement of the heat elimination $\alpha(t)$.

The results obtained permit establishment of the main source of error in the modeling and its significance in the cases considered. For the model (1) the influence of installing the measurement system along the rod axis is minimal, while a local change in the continuity of the specimen for the models (5)-(7) causes a certain difference in the coefficient α_1 from the analogous value obtained by handbook data. This distinction grows with detailing of the description of the thermal state performed within the framework of the model of a continuous rod. Among the other methodological errors, the absence of taking account of the nonlinearity in the thermophysical properties and the inhomogeneity of the heat distribution can be taken into account. But the satisfactory consistency with observations and the compatibility of the estimates of parameters of the same kind, obtained in different identification stages, show that the errors of the models used do not cause significant errors in the quantities found.

Therefore, the theory developed for inverse problems can be extended to the case of absence of a sufficient quantity of a priori information about the form of the adequate model of the process. For such situations, the joint solution of a whole set of inverse problems in which a conception reflecting the proposed structure of the effective factors is included, turns out to be effective. Its operator mode can be given by a family of differential equations, say, whose composition is determined by information about the qualitative properties of the object of investigation. An estimate of the identification error because of modeling errors can be made on the basis of a comparative analysis of the results of solving different inverse problems obtained by using the very same initial data. The method proposed for processing the results of experiment permitted realization of an estimate of the state of the object and the establishment of the significance of the effective factors within the limited volume of the observation sample.

NOTATION

u, temperature field; c, specific heat; ρ, density; α, heat elimination; ε, emissivity; σ, Stefan–Boltzmann constant; u_{av}, temperature of the environment; u_0, initial temperature distribution; $\theta_{0,1}$, boundary temperatures; T, upper bound of the observation time; R, rod radius; l, estimation segment; r_T, φ_T, z_T, coordinates of the temperature measurement point; n, sample volume; and f, g, h, q, s, auxiliary quantities.

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